On the Galerkin method for semilinear parabolic-ordinary systems

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Abstract

We consider a general system of n_1 semilinear parabolic partial differential equations and n_2 ordinary differential equations, with locally Lipschitz continuous nonlinearities. We analyse the well-posedness of this problem, exploiting the tools of the semigroups theory, and derive other further regularity results and conditions for the boundedness of the solution.

We define the Galerkin semidiscrete approximation to the system and derive optimal order error estimates in L^2 norm, under various assumptions on the nonlinear terms, on the finite dimensional subspaces in which the approximation is sought and on the regularity of the exact solution. As a byproduct, we can also show that the approximate solution is globally defined and bounded.

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1 Semilinear parabolic-ordinary systems

In this work we study a general class of reaction-diffusion problems, described by systems of weakly coupled semilinear parabolic and ordinary differential equations, with Neumann boundary conditions and Lipschitz continuous nonlinearities. Examples of such systems can be found in electrocardiology [1], [2] and have already been studied analytically by Pao [4].

Let Ω be a bounded open set in $\mathbb{R}^n$, $n = 1, 2, 3$ and let the boundary $\partial\Omega$ be $C^{2+\alpha}$, $\alpha \in (0, 1)$. Let's indicate by $\nu = (\nu_1, \dots, \nu_n)^T$ the outward normal to $\partial\Omega$. Let's consider the following semilinear system of n_1 parabolic equations and n_2 ordinary differential equations, with nonhomogeneous Neumann boundary conditions

$$\begin{array}{lll} \partial_t u_i + \mathcal{L}_i u_i = g_i(x,t,\mathbf{u}) & in\ \Omega\times]0,T] & (i=1,\dots,n_1) \\ \partial_t u_i = g_i(x,t,\mathbf{u}) & in\ \Omega\times]0,T] & (i=n_1+1,\dots,n_1+n_2) \\ \mathcal{B}_i u_i = q_i(x,t) & in\ \partial\Omega\times]0,T] & (i=1,\dots,n_1) \\ u_i(x,0) = u_{i,0}(x) & in\ \Omega & (i=1,\dots,n_1+n_2) \end{array} \tag{1}$$

where $\mathcal{L}_i$ and $\mathcal{B}_i$, $i=1,\dots,n_1$, are operators of the form

$$\mathcal{L}_i w := -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}^{(i)}(x)\frac{\partial w}{\partial x_k}\right) + \sum_{j=1}^{n} b_j^{(i)}(x)\frac{\partial w}{\partial x_j} + a_0^{(i)}(x)w, \tag{2}$$

$$\mathcal{B}_i w := \frac{\partial w}{\partial \nu_{\mathcal{L}}} = \sum_{j,k=1}^{n} a_{jk}^{(i)}(x)\frac{\partial w}{\partial x_k}\nu_j; \tag{3}$$

the matrices $\left(a_{jk}^{(i)}\right)$ are symmetric and the operators $\mathcal{L}_i$ are uniformly elliptic in $\overline{\Omega}$; $b_j^{(i)}, a_0^{(i)} \in C(\overline{\Omega})$, $a_{jk}^{(i)} \in C^1(\overline{\Omega})$ and the functions $q_i(x,t) \in C^{1+\alpha}([0,T];C^{1+\alpha}(\partial\Omega))$. The ellipticity of $\mathcal{L}_i$ implies that the boundary conditions are of *non tangential* type, i.e.

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{jk}^{(i)} n_j\right) n_k \neq 0, \qquad \forall x \in \partial\Omega.$$

Let's consider first the homogeneous case $q_i \equiv 0$, $i=1,2,\dots,n_1$.

The well-posedness of problem (1) can be reduced to the study of an abstract evolution equation in the space $X = C(\overline{\Omega})^{n_1+n_2}$, by defining the *realization of* $-\mathcal{L}_i$ *in* $X_i = C(\overline{\Omega})$ *with homogeneous first order oblique boundary conditions*

$$D(A_i) := \{u \in \bigcap_{p\geq 1} W^{2,p}(\Omega) : \mathcal{L}_i u \in C(\overline{\Omega}),\ \mathcal{B}_i u = 0 \text{ on } \partial\Omega\}, \qquad A_i u = -\mathcal{L}_i u.$$

We recall that, for $n=1$, $D(A_i) = \{u \in C^2(\overline{\Omega}) :\ \mathcal{B}_i u_{|\partial\Omega} = 0\}$ and that (see [3] for the general theory), for every $i=1,\dots,n_1$, the *resolvent set* $\rho(A_i)$ and the *resolvent operator* $R(\lambda,A_i)$ are defined as $\rho(A_i) := \{\lambda \in \mathbb{C} :\ \exists(\lambda I - A_i)^{-1} \in L(X_i)\}$ and $R(\lambda,A_i) := (\lambda I - A_i)^{-1}, \qquad \forall\lambda \in \rho(A_i)$, where $L(X_i)$ is the space of the continuous linear functionals of the Banach space X_i.

Definition 1.1 *Let X be a Banach space, with norm $\| \cdot \|_X$. A linear operator $A : D(A) \subset X \to X$ is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in]\pi/2, \pi[$, $M > 0$ such that*

$$\begin{cases} (i) \quad \rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \ \lambda \neq \omega, \ |\arg(\lambda - \omega)| < \theta\}, \\ (ii) \quad \|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda - \omega|} \ \forall \lambda \in S_{\theta,\omega}. \end{cases}$$

Given a sectorial operator A in X, we define the intermediate spaces between X and $D(A)$ $(0 < \alpha < 1)$ as $D_A(\alpha, \infty) := \{x \in X : \ t \mapsto v(t) = \|t^{1-\alpha} A e^{tA} x\|_X \in L^\infty(0,1)\}$, endowed with the norm $\|x\|_{D_A(\alpha,\infty)} = \|x\|_X + \|v\|_{L^\infty(0,1)}$. It can be proved [3] that $D(A) \subset D_A(\alpha, \infty) \subset \overline{D(A)}$, $0 < \alpha < 1$.

In the case of (2) and (3), it can be shown that A_i is sectorial in X_i and that

$$\overline{D(A_i)} = C(\overline{\Omega}), \qquad D_{A_i}(\alpha, \infty) = \begin{cases} C^{2\alpha}(\overline{\Omega}), & \text{if } \alpha < 1/2, \\ C^{2\alpha}_{\mathcal{B}_i}(\overline{\Omega}), & \text{if } \alpha > 1/2, \end{cases}$$

where $C^{2\alpha}_{\mathcal{B}_i}(\overline{\Omega})$ is the subspace of $C^{2\alpha}(\overline{\Omega})$ whose functions φ satisfy $\mathcal{B}_i \varphi = 0$ on $\partial\Omega$.

To simplify the exposition, we consider the case of a system of the form (1) with $n_1 = 1$ and $n_2 = 2$. The following results can be easily extended to the general case.

Let's define the operator A in the space X

$$D(A) = D(A_1) \times C(\overline{\Omega}) \times C(\overline{\Omega}), \qquad A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where 0 represents the *null operator*. The following theorem can be easily verified

Theorem 1.2 *The operator A is sectorial in X and $\overline{D(A)} = X$. Moreover $D_A(\alpha, \infty) = D_{A_1}(\alpha, \infty) \times C(\overline{\Omega})^2$, $0 < \alpha < 1$.*

Set in (1)

$$u(t)(x) := \mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))^T,$$

$$f(t,u)(x) := (g_1(x,t,\mathbf{u}(x)), g_2(x,t,\mathbf{u}(x)), g_3(x,t,\mathbf{u}(x)))^T,$$

$$u_0 := (u_{1,0}(x), u_{2,0}(x), u_{3,0}(x))^T.$$

We get the (semilinear parabolic) abstract evolution equation

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)) & t \in]0, T] \\ u(0) = u_0. \end{cases} \tag{4}$$

Local existence and uniqueness results for the abstract problem (4) are stated by the following [3]

Theorem 1.3 *Let $A : D(A) \subset X \to X$ be a sectorial operator and $f : [0,T] \times X \to X$ be a continuous function locally Lipschitz continuous with respect to u, i.e. $\forall R > 0$ $\exists L = L(R) > 0$ such that*

$$\|f(t,u) - f(t,v)\|_X \leq L\|u - v\|_X, \qquad \forall t \in [0,T] \quad \forall u, v \in B(0,R), \tag{5}$$

where $B(0,R)$ is the ball centred in 0 with ray R in X. Moreover suppose that there exists $\alpha \in (0,1)$ such that for every $R > 0$

$$\|f(t,v) - f(s,v)\|_X \leq C(R)(t-s)^\alpha, \qquad 0 \leq s \leq t \leq T, \ \|v\|_X \leq R. \tag{6}$$

The following statements hold:

(i) if $u_0 \in \overline{D(A)}$, then (4) has a unique maximal classical solution $u \in C([0,T'[;X) \cap C(]0,T'[;D(A)) \cap C^1(]0,T'[;X)$, $0 < T' \leq T$;

(ii) if $u_0 \in D(A)$ and $Au_0 + f(0,u_0) \in \overline{D(A)}$, then u is a maximal strict solution of (4), i.e. $u \in C([0,T'[;D(A)) \cap C^1([0,T'[;X)$;

(iii) if $u_0 \in D(A)$ and $Au_0 + f(0,u_0) \in D_A(\alpha,\infty)$ for some $\alpha \in (0,1)$, then for every $b < T'$ we have $u', Au \in C^\alpha([0,b];X)$, $u' \in B([0,b];D_A(\alpha,\infty))$, where $B([a,b];Y)$ is the space of the bounded functions: $[a,b] \to Y$.

Among all the possible global existence results, we only recall the following [3]

Theorem 1.4 *Suppose there exists $C > 0$ such that*

$$\|f(t,u)\|_X \leq C(1 + \|u\|_X), \quad \forall u \in X, \ \forall t \in [0,T]. \tag{7}$$

Let $u : \ [0,T'[\to X$ be a local solution of (4). Then u is bounded in $[0,T'[$ with values in X and therefore it is a global solution of (4).

Now, let us consider again the concrete problem (1). Suppose the functions g_i, $i = 1,2,3$, are continuous and for $0 < \alpha < 1$ satisfy

for every $R > 0$ there exist some positive constants $K_i = K_i(R)$ such that

$$|g_i(x,t,\mathbf{u}) - g_i(x,s,\mathbf{w})| \leq K_i(|t-s|^\alpha + |\mathbf{u} - \mathbf{w}|_1) \ \ \text{for } x \in \overline{\Omega}, \ t,s \in [0,T], \ |\mathbf{u}|_1, |\mathbf{w}|_1 \leq R, \tag{8}$$

where $|\mathbf{u}|_1 = \sum_{i=1}^3 |u_i|$.

A direct application of the abstract theorems 1.3 and 1.4 to the reaction-diffusion problem (1), via its abstract formulation (4), yields the following theorems. See [6] for the details of the proof.

Theorem 1.5 *Under condition (8), the following statements hold:*

(i) if $u_{i,0} \in C(\overline{\Omega})$, for $i = 1,2,3$, then (1) with $q_i = 0$ has a unique maximal classical solution $\mathbf{u}$ continuous in $\overline{\Omega} \times [0,T'[$, with $\mathbf{u}$ differentiable with respect to t in $\overline{\Omega}\times]0,T'[$ and $u_1(\cdot,t) \in W^{2,p}(\Omega)$ for every $p \geq 1$ and $0 < t < T' < T$;

(ii) if $u_{1,0} \in \bigcap_{p\geq 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0} = 0$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ and $\mathcal{L}_1 u_{1,0} \in C(\overline{\Omega})$, then $\mathbf{u}$ is a maximal strict solution of (1), i.e. $(u_i)_t$, $\mathcal{L}_1 u_1 \in C(\overline{\Omega} \times [0,T'[)$, for $i = 1,2,3$;

(iii) if $u_{1,0} \in \bigcap_{p\geq 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0} = 0$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ and $-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0) \in C^{2\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1/2)$, or $-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0) \in C^{2\alpha}_{\mathcal{B}_1}(\overline{\Omega})$, for some $\alpha \in (1/2,1)$, then $\partial_t u_1$ belongs to $C^{2\alpha,\alpha}(\overline{\Omega}\times[0,b])$, $\mathcal{A}_1 u_1$ belongs to $C^{0,\alpha}(\overline{\Omega}\times[0,b])$, for every $b < T'$. In particular, if $\alpha > 1/2$ then $(u_1)_{x_i t}$ is continuous for $i = 1,\ldots,n$.

Theorem 1.6 *Suppose all the hypothesis of Theorem 1.5 hold. If the nonlinear terms satisfy*

$$\exists C > 0 \ s.\ t. \quad \sum_{i=1}^{3} |g_i(x,t,\mathbf{u})| \leq C(1+|\mathbf{u}|_1), \qquad \forall t \in [0,T],\ \forall x \in \overline{\Omega},\ \forall \mathbf{u} \in \mathbb{R}^3, \tag{9}$$

$i = 1,2,3$, then the solution $\mathbf{u}$ is global.

In the case of nonhomogeneous boundary conditions, the abstract theory can not be applied directly, since $D(A_1)$ is endowed with homogeneous boundary conditions. Usually, in these cases, it is useful to consider the boundary terms q_i as traces at $\partial\Omega \times [0,T]$ of $\mathcal{B}_i Q_i$, where Q_i are sufficiently smooth functions defined in $\overline{\Omega} \times [0,T]$, so that the nonhomogeneous problem can be reduced to a homogeneous one in the unknowns $\tilde{u}_1 = u_1 - Q_1$ and $\tilde{u}_i = u_i$, $i = 2,3$, and then the previous Theorem 1.5 can be applied. More precisely, we can set

$$Q_i(\cdot,t) = \mathcal{N}_i q_i(\cdot,t), \qquad 0 \leq t \leq T,$$

where $\mathcal{N}_i$ is an operator which belongs to $L(C^\theta(\partial\Omega), C^{\theta+1}(\overline{\Omega}))$ for every $\theta \in [0,1]$ (for $\theta \in [0,1+\alpha]$ if $\partial\Omega$ is uniformly $C^{2+\alpha}$, $0 \leq \alpha < 1$) such that

$$\mathcal{B}_i(\mathcal{N}_i f)_{|\partial\Omega} = f, \quad \forall f \in C(\partial\Omega).$$

Then the following theorem holds (see [6] for further details of the proof).

Theorem 1.7 *Suppose that $\partial\Omega$ is uniformly $C^{2+\alpha}$ for some $\alpha \in (0,1)$, that (8) holds and that $q_1 \in C^{1+\alpha}([0,T]; C^{1+\alpha}(\partial\Omega))$. The following statements hold:*

(i) if $u_{i,0} \in C(\overline{\Omega})$, for $i = 1,2,3$, then (1) has a unique maximal classical solution $\mathbf{u}$ continuous in $\overline{\Omega} \times [0,T'[$, with $\mathbf{u}$ differentiable with respect to t in $\overline{\Omega}\times]0,T'[$ and $u_1(\cdot,t) \in W^{2,p}(\Omega)$ for every $p \geq 1$ and $0 < t < T' < T$;

(ii) if $u_{1,0} \in \bigcap_{p\geq 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0}(x) = q_1(x,0)$ for $x \in \partial\Omega$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ and $\mathcal{L}_1 u_{1,0} \in C(\overline{\Omega})$, then $\mathbf{u}$ is a maximal strict solution of (1), i.e. $(u_i)_t$, $\mathcal{L}_1 u_1 \in C(\overline{\Omega} \times [0,T'[)$, for $i = 1,2,3$;

(iii) if $u_{1,0} \in \bigcap_{p\geq 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0}(x) = q_1(x,0)$ for $x \in \partial\Omega$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ and $-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0) \in C^{2\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1/2)$, or $-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0) \in C^{2\alpha}(\overline{\Omega})$ and $\mathcal{B}_1(-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0)) = \partial_t q_1(\cdot,0)$ on $\partial\Omega$, for some $\alpha \in (1/2,1)$, then $\partial_t u_1$ belongs to $C^{2\alpha,\alpha}(\overline{\Omega} \times [0,b])$, $\mathcal{A}_1 u_1$ belongs to $C^{0,\alpha}(\overline{\Omega} \times [0,b])$, for every $b < T'$.

2 Galerkin approximation

Let us now introduce the Galerkin semidiscretization of system (1). The convergence of such method is studied according to the smoothness of the solution of (1) and to the approximation properties of the finite dimensional spaces V_h used.

Let's indicate by $\mathcal{A}^{(i)}(.,.)$ the bilinear form associated to the operator $\mathcal{L}_i$

$$\mathcal{A}^{(i)}(w,v) := \int_\Omega \left[\sum_{j,k=1}^{n} a_{jk}^{(i)} \frac{\partial w}{\partial x_k} \frac{\partial v}{\partial x_j} + \sum_{j=1}^{n} b_j^{(i)} v \frac{\partial w}{\partial x_j} + a_0^{(i)} wv \right] dx.$$

We can suppose without restriction that the bilinear forms $\mathcal{A}^{(i)}(.,.)$ are continuous and coercive in $H^1(\Omega)$, i.e. there exist some constants $\alpha^{(i)} > 0$ such that

$$\mathcal{A}^{(i)}(v,v) \geq \alpha^{(i)} \|v\|_1^2 \qquad \forall v \in H^1(\Omega). \tag{10}$$

Let us consider again the case $n_1 = 1$ and $n_2 = 2$, leaving to the reader the extension to the general case. Suppose that the data are such that (1) has a unique bounded soluton $\mathbf{u} = (u_1, u_2, u_3)^T$, which is sufficiently smooth for all our purposes. Therefore, under the hypothesis specified in Section 1, let us consider the following weak formulation of problem (1)

$$\text{find } u_1 \in L^2(0,T; H^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)),\ u_2, u_3 \in C^0([0,T]; L^2(\Omega))$$

$$\text{with } \operatorname{supess}_{\Omega\times]0,T[} |u_i| < \infty,\ i = 1,2,3 \ \text{ such that}$$

$$\frac{d}{dt}(u_1(t),\varphi)+\mathcal{A}^{(1)}(u_1(t),\varphi) = (g_1(x,t,u_1(t),u_2(t),u_3(t)),\varphi)+(q_1(t),\varphi)_{\partial\Omega}, \quad \forall\varphi\in H^1(\Omega)$$
$$\frac{d}{dt}(u_i(t),\psi) = (g_i(x,t,u_1(t),u_2(t),u_3(t)),\psi), \quad \forall\psi\in L^2(\Omega) \quad i=2,3$$
$$u_i(0)=u_{i,0}, \quad i=1,2,3. \tag{11}$$

Let us define the Galerkin semidiscretization of (1). Fix the families of subspaces $\{V_h\}_{h>0}$ of $H^1(\Omega)$ and $\{W_h\}_{h>0}$ of $L^2(\Omega)$, with finite dimension N_h, and define the Galerkin semidiscrete problem

$$\text{find } u_1 :]0,T[\to V_h,\ u_2,u_3 :]0,T[\to W_h, \text{ s. t. } \forall\varphi_h\in V_h\ \forall\psi_h\in W_h$$
$$\frac{d}{dt}(u_{1,h}(t),\varphi_h)+\mathcal{A}^{(1)}(u_{1,h}(t),\varphi_h) = (g_1(x,t,u_{1,h}(t),u_{2,h}(t),u_{3,h}(t)),\varphi_h)+(q_1(t),\varphi_h)_{\partial\Omega},$$
$$\frac{d}{dt}(u_{i,h}(t),\psi_h) = (g_i(x,t,u_{1,h}(t),u_{2,h}(t),u_{3,h}(t)),\psi_h), \quad i=2,3$$
$$u_{i,h}(0)=u_{i,0,h}, \quad i=1,2,3, \tag{12}$$

where $u_{1,0,h}\in V_h$ and $u_{i,0,h}\in W_k$, for $i=2,3$, are suitable approximations of the initial data.

In this section we study the properties of the semidiscrete system (12). The proof of the convergence of the semidiscrete solution to $\mathbf{u}$ in the presence of locally Lipschitz continuous nonlinear terms needs some L^∞ estimates on the semidiscrete solution to be provided, in order to assure it doesn't blow up. Therefore, we first analyse the case of globally Lipschitz continuous reaction terms. It must be noted that, in both cases, the obtained results are independent of the particular choice of the spaces V_h and W_h, since they are based only on the smoothness of the solution $\mathbf{u}$ and on the approximation properties of such spaces.

Let us now consider the case of functions g_i, $i=1,2,3$, globally Lipschitz continuous with respect to the variable $\mathbf{u}$, i.e. we suppose there exist some constants $K_i>0$ such that

$$|g_i(x,t,\mathbf{u})-g_i(x,t,\mathbf{w})| \le K_i|\mathbf{u}-\mathbf{w}|_1 \quad \forall\mathbf{u},\mathbf{w}\in\mathbb{R},\ x\in\overline{\Omega},\ t\in[0,T]. \tag{13}$$

In this case, the global Lipschitz continuity of g_i yields the global Lipschitz continuity of the right hand side of system (12) and hence, by standard existence theorems for these systems, the semidiscrete solution is globally defined. Let us indicate by $\mathbf{u}_h=(u_{1,h},u_{2,h},u_{3,h})^T$ the unique solution of (12); note that the solution $\mathbf{u}=(u_1,u_2,u_3)^T$ of (1) is also solution to (11).

In the following we suppose that the family $\{V_h\}$ satisfies the approximation property

$$\lim_{h\to 0} \inf_{\varphi_h \in L^\infty(0,T;V_h)} \sup_{[0,T]} h^j \|\varphi(t) - \varphi_h(t)\|_j = 0 \quad \forall \varphi \in L^\infty(0,T;H^1(\Omega)), \tag{14}$$

where $j = 0, 1$; while for the family $\{W_h\}$ the following property holds

$$\lim_{h\to 0} \inf_{\psi_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|\psi(t) - \psi_h(t)\|_0 = 0 \quad \forall \psi \in S, \tag{15}$$

where $S := L^\infty(0,T;L^2(\Omega)) \cap \{f : \ f(t) \in C_0^\infty(\Omega) \text{ a.e. } t \in [0,T]\}$. We state, without proof [6], the following

Lemma 2.1 *Let $\{W_h\}$ be a family of subspaces of $L^2(\Omega)$ satisfying (15). For every $w \in C^0([0,T];L^2(\Omega))$ we have*

$$\lim_{h\to 0} \inf_{w_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|w(t) - w_h(t)\|_0 = 0.$$

We have the following convergence result in the L^2-norm.

Theorem 2.2 *Suppose that g_i, $i = 1, 2, 3$, satisfies the condition (13) and that the solution of (11) is such that $u_1 \in C^0([0,T];L^2(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega))$ and $u_2, u_3 \in C^1([0,T];L^2(\Omega))$. If the families $\{V_h\}$ and $\{W_h\}$ satisfy (14) and (15) respectively, then the solution of (12) exists for $t \leq T$ and converges in the L^2-norm to the solution $\mathbf{u}$ of (11).*

The analysis of the error between $\mathbf{u}_h$ and $\mathbf{u}$ can be accomplished comparing the Galerkin solution $u_{1,h}(t)$ to the *elliptic projection* $W_1(t)$ of the solution $u_1(t)$ of (11) onto V_h, defined for a.e. $t \in]0,T[$ by

$$\mathcal{A}^{(1)}(W_1(t), \varphi_h) = \mathcal{A}^{(1)}(u_1(t), \varphi_h) \qquad \forall \varphi_h \in V_h. \tag{16}$$

As it is well known, the error $\rho_1(t) = u_1(t) - W_1(t)$ satisfies for $t \leq T$ and for some $\bar{C} > 0$

$$\|\rho_1(t)\|_0 \leq \bar{C} h \inf_{\varphi_h \in V_h} \|u_1(t) - \varphi_h\|_1. \tag{17}$$

The rest of the error in v_h is then $\theta_1(t) = u_{1,h}(t) - W_1(t) \in V_h$.

In the following C will denote constants, not necessarily the same at different occurrences, which are independent of h and the functions involved. Similarly, c will denote constants which are independent of h but which may depend on the solution $\mathbf{u}(t)$ of (11). We shall assume that the initial values are chosen in such a way that

$$\|u_{1,0,h} - W_1(0)\|_0 \le ch^{\mu}, \ \|u_{2,0,h} - u_{2,0}\|_0 \le ch^{\mu'}, \ \|u_{3,0,h} - u_{3,0}\|_0 \le ch^{\mu'}, \tag{18}$$

for some $\mu, \mu' > 0$, where $W_1(t)$ is defined by (16).

Proof. Let us subtract (11) from (12), for fixed t. Taking into account (16), setting $\varphi = \varphi_h = \theta_1(t)$ and using Cauchy-Schwarz's inequality, the coerciveness of $\mathcal{A}^{(1)}$, Lipschitz condition for g_1 and Young's inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, we obtain for a.e. $t < T$

$$\frac{1}{2}\frac{d}{dt}\|\theta_1\|_0^2 \le \frac{1}{2}\|(\rho_1)_t\|_0^2 + \frac{1}{2}(5 + K_1^2)\|\theta_1\|_0^2 + \frac{K_1^2}{2}\|\rho_1\|_0^2 + \frac{K_1^2}{2}\|U_2\|_0^2 + \frac{K_1^2}{2}\|U_3\|_0^2,$$

where $U_2(t) = u_{2,h}(t) - u_2(t)$ and $U_3(t) = u_{3,h}(t) - u_3(t)$ for $t \in [0, T]$. Since time differentiation commutes with elliptic projection, by (17) we have $\|(\rho_1)_t\|_0 \le \bar{C}h \inf_{\varphi_h \in V_h} \|(u_1)_t(t) - \varphi_h\|_1$; hence

$$\frac{1}{2}\frac{d}{dt}\|\theta_1\|_0^2 \le C_1\|\theta_1\|_0^2 + C_2\|U_2\|_0^2 + C_2\|U_3\|_0^2 + C_3h^2\|u_1 - \varphi_h\|_1^2 + C_4h^2\|(u_1)_t - \bar{\varphi}_h\|_1^2, \tag{19}$$

where $C_1 = (5 + K_1^2)/2 > 0$, $C_2 = K_1^2/2 > 0$, $C_3 = C_2\bar{C}^2$, $C_4 = \bar{C}^2/2$ and φ_h, $\bar{\varphi}_h$ are arbitrary elements of V_h.

Now, let us consider the first ordinary differential equation, in its weak form; we have for $t < T$

$$((U_2)_t(t), \psi_h) = (g_2(x, t, u_{1,h}, u_{2,h}, u_{3,h}) - g_2(x, t, u_1, u_2, u_3), \psi_h) \qquad \forall \psi_h \in W_h. \tag{20}$$

Set $\psi_h = u_{2,h}(t) - w_h \in W_h$, where w_h is an arbitrary element of W_h. As before, this yields

$$\frac{1}{2}\frac{d}{dt}\|U_2\|_0^2 \le C_5\|U_2\|_0^2 + \frac{1}{2}\|(U_2)_t\|_0^2 + \frac{5}{2}\|u_2 - w_h\|_0^2 + K_2^2\|\theta_1\|_0^2 + K_2^2\|\rho_1\|_0^2 + K_2^2\|U_3\|_0^2, \tag{21}$$

with $C_5 = 2 + K_2^2$. Moreover, choosing $(u_{2,h})_t - \phi_h \in W_h$ in (20), where ϕ_h is an arbitrary element of W_h, we have for every $t \in]0, T[$

$$\|(U_2)_t\|_0^2 \le \|(u_2)_t - \phi_h\|_0^2 + C_6\|\theta_1\|_0^2 + C_6\|\rho_1\|_0^2 + C_6\|U_2\|_0^2 + C_6\|U_3\|_0^2,$$

where $C_6 = 10K_2^2$. The last inequality can be combined with (21) yielding for $t < T$

$$\frac{1}{2}\frac{d}{dt}\|U_2\|_0^2 \le \frac{1}{2}\|(u_2)_t - \phi_h\|_0^2 + \frac{5}{2}\|u_2 - w_h\|_0^2 + C_5'\|U_2\|_0^2 + C_6'\|\theta_1\|_0^2 + C_6'\|\rho_1\|_0^2 + C_6'\|U_3\|_0^2, \tag{22}$$

for suitable constants C'_5 e C'_6.

Similarly, the second ordinary differential equation yields the inequality

$$\frac{1}{2}\frac{d}{dt}\|U_3\|_0^2 \leq \frac{1}{2}\|(u_3)_t - \bar{\phi}_h\|_0^2 + \frac{5}{2}\|u_2 - \bar{w}_h\|_0^2 + C'_7\|U_3\|_0^2 + C'_8\|\theta_1\|_0^2 + C'_8\|\rho_1\|_0^2 + C'_8\|U_2\|_0^2, \tag{23}$$

for $\bar{\phi}_h, \bar{w}$ arbitrary elements of W_h and for suitable constants $C'_7, C'_8 > 0$.

Now, we sum up (19), (22) and (23) and integrate over $[0,t]$, $t < T$; using (17), for a suitable constant C we get

$$\|\theta_1(t)\|_0^2 + \|U_2(t)\|_0^2 + \|U_3(t)\|_0^2 \leq \|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$

$$+\int_0^t [\|(u_2)_t(s) - \phi_h\|_0^2 + 5\|u_2(s) - w_h\|_0^2 + \|(u_3)_t(s) - \bar{\phi}_h\|_0^2 + 5\|u_3(s) - \bar{w}_h\|_0^2 + Ch^2\|u_1(s) - \varphi_h\|_1^2 +$$

$$+Ch^2\|(u_1)_t(s) - \bar{\varphi}_h\|_1^2]ds + C\int_0^t \left(\|\theta_1(s)\|_0^2 + \|U_2(s)\|_0^2 + \|U_3(s)\|_0^2\right) ds,$$

where $\varphi_h, \bar{\varphi}_h \in L^\infty(0,T;V_h)$ and $\phi_h, \bar{\phi}_h, w_h, \bar{w}_h \in L^\infty(0,T;W_h)$ are arbitrary elements.

Gronwall's lemma yields $\forall t < T$

$$\|\theta_1(t)\|_0^2 + \|U_2(t)\|_0^2 + \|U_3(t)\|_0^2 \leq [\|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$

$$+\int_0^T (\|(u_2)_t(s) - \phi_h(s)\|_0^2 + 5\|u_2(s) - w_h(s)\|_0^2 + \|(u_3)_t(s) - \bar{\phi}_h(s)\|_0^2 + 5\|u_3(s) - \bar{w}_h(s)\|_0^2 +$$

$$+Ch^2\|u_1(s) - \varphi_h(s)\|_1^2 + Ch^2\|(u_1)_t(s) - \bar{\varphi}_h(s)\|_1^2)\; ds]e^{CT},$$

and hence

$$\|\mathbf{u}_h(t) - \mathbf{u}(t)\|_0^2 \leq 2[\|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$

$$+\int_0^T (\|(u_2)_t(s) - \phi_h(s)\|_0^2 + 5\|u_2(s) - w_h(s)\|_0^2 + \|(u_3)_t(s) - \bar{\phi}_h(s)\|_0^2 + 5\|u_3(s) - \bar{w}_h(s)\|_0^2 +$$

$$+Ch^2\|u_1(s) - \varphi_h(s)\|_1^2 + Ch^2\|(u_1)_t(s) - \bar{\varphi}_h(s)\|_1^2)\; ds]e^{CT} + 2\bar{C}h^2 \inf_{\varphi_h \in V_h} \|u_1(t) - \varphi_h\|_1^2. \tag{24}$$

The arbitrariness of the test functions implies

$$\|\mathbf{u}_h(t) - \mathbf{u}(t)\|_0^2 \leq 2e^{CT}[\|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$

$$+T \inf_{\phi_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|(u_2)_t - \phi_h\|_0^2 + 5T \inf_{w_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|u_2 - w_h\|_0^2 +$$

$$+T \inf_{\bar{\phi}_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|(u_3)_t - \bar{\phi}_h\|_0^2 + 5T \inf_{\bar{w}_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|u_3 - \bar{w}_h\|_0^2 +$$

$$+CTh^2 \inf_{\varphi_h \in L^\infty(0,T;V_h)} \sup_{[0,T]} \|u_1 - \varphi_h\|_1^2 + CTh^2 \inf_{\bar{\varphi}_h \in L^\infty(0,T;V_h)} \sup_{[0,T]} \|(u_1)_t - \bar{\varphi}_h\|_1^2] +$$

$$+2\bar{C}h^2 \inf_{\varphi_h \in V_h} \|u_1(t) - \varphi_h\|_1^2,$$

and, from (14),(15), (18) and from Lemma 2.1, we have the convergence as $h \to 0$. □

In order to have an optimal error estimate in L^2-norm, we have to introduce further assumptions on the spaces V_h and W_h, and hypothesis on the spatial smoothness of the solution.

Definition 2.3 *A family of spaces $\{X_h\}$ is said to be of class $\mathcal{S}_{k,\mu}$ (with $k \le \mu$) if $X_h \subset H^k(\Omega)$ and there exists a constant $C > 0$ such that*

$$\forall w \in H^\mu(\Omega), \qquad \inf_{w_h \in X_h} \sum_{j=0}^{k} h^j \|w - w_h\|_j \le Ch^\mu \|w\|_\mu.$$

Since we are dealing with a second order partial differential system coupled with ordinary differential equations, in the sequel we shall assume that $\{V_h\}$ is of class $\mathcal{S}_{1,\mu}$ ($\mu \ge 2$) and $\{W_h\}$ is of class $\mathcal{S}_{0,\mu'}$ ($\mu' \ge 1$).

We have the following convergence result

Theorem 2.4 *Suppose that g_i, $i = 1, 2, 3$, satisfies the Lipschitz condition (13), that the family $\{V_h\}$ is of class $\mathcal{S}_{1,\mu}$ ($\mu \ge 2$) and $\{W_h\}$ of class $\mathcal{S}_{0,\mu'}$ ($\mu' \ge 1$) and that (18) holds. Assume that the solution $\mathbf{u}$ is such that $u_1 \in H^1(0,T;H^\mu(\Omega))$ and $u_2, u_3 \in H^1(0,T;H^{\mu'}(\Omega))$. Then we have the following error estimate*

$$\|\mathbf{u}_h - \mathbf{u}\|_0 \le ch^{\min(\mu,\mu')}, \qquad \forall t \in [0,T],$$

where c depends on the constants K_i and on the solution $\mathbf{u}$.

Proof. The thesis follows easily from the error estimate (24) and from the assumptions on the spaces V_h and W_h. □

Let us move to the case of locally Lipschitz continuous functions g_i. In this case the proof of the convergence needs further assumptions on the families $\{V_h\}$ and $\{W_h\}$, since in general local Lipschitz continuity of the g_i's does not guarantee global existence of the solution to (12) either and therefore we have to provide estimates which assure that the semidiscrete solution does not blow up. The idea of the proof was used by

Thomèe [7] for general semilinear equations of parabolic type and extends to a system of the form (1) what proved in [5].

In the sequel we will assume that the families $\{V_h\}$ and $\{W_h\}$ are of class $\mathcal{S}_{1,\mu}$ and $\mathcal{S}_{1,\mu'}$ respectively and satisfy the following *inverse inequality*

$$\|\varphi_h\|_\infty \leq C_1 h^{-\nu} \|\varphi_h\|_0 \quad \forall \varphi_h \in V_h,\ h \leq h_1, \quad \text{for some } \nu \text{ and } h_1, \tag{25}$$

$$\|\psi_h\|_\infty \leq C_2 h^{-\nu'} \|\psi_h\|_0 \quad \forall \psi_h \in W_h,\ h \leq h_2, \quad \text{for some } \nu' \text{ and } h_2. \tag{26}$$

We start by observing that, for fixed h, system (12) has a unique local solution in $[0, T_h]$, for some $T_h \leq T$. In the analysis developed below, we shall show that, under suitable assumptions on $\{V_h\}$ and $\{W_h\}$ and for h sufficiently small, T_h can be taken to be equal to T. Hence, global solvability of system (12) follows. This can be achieved by providing an estimate of the error $\mathbf{u}_h - \mathbf{u}$ in the L^∞-norm, which allows to derive boundedness of the semidiscrete solution $\mathbf{u}_h$. Therefore, assume that the following approximation properties hold

$$\lim_{h\to 0} \sup_{[0,T]} \inf_{w_h \in V_h} \{\|w(t) - w_h\|_\infty + h^{-\nu}\|w(t) - w_h\|_0\} = 0 \quad \forall w \in L^\infty(0, T; H^\mu(\Omega)), \tag{27}$$

$$\lim_{h\to 0} \sup_{[0,T]} \inf_{w_h \in W_h} \{\|w(t) - w_h\|_\infty + h^{-\nu'}\|w(t) - w_h\|_0\} = 0 \quad \forall w \in L^\infty(0, T; H^{\mu'}(\Omega)), \tag{28}$$

for some $\nu, \nu' > 0$. Let us indicate by $\mathbf{u}_h = (u_{1,h}, u_{2,h}, u_{3,h})$ the unique maximal local solution of (12). Moreover, let Σ be the range of the solution $\mathbf{u}$. Let us fix $\delta > 0$ sufficiently large as to include the initial datum $\mathbf{u}_{0,h} = (u_{1,0,h}, u_{2,0,h}, u_{3,0,h})$ in a closed neighborhood Σ_δ of Σ and let $K_i > 0$ be constants such that the Lipschitz condition (8) holds in Σ_δ.

Heuristically we might argue that, since the approximate solution $\mathbf{u}_h$ is always going to be close to $\mathbf{u}$, it belongs to Σ_δ. In order to show that this is the case, we have to provide maximum norm estimates for the approximation error, since closeness in the sense of L^2 or H^1 does not automatically imply that $\mathbf{u}_h$ belongs to Σ_δ for small h.

Now we state the main theorem of this section

Theorem 2.5 *Assume that g_i, $i = 1, 2, 3$, satisfies the Lipschitz condition (8). Let $\{V_h\}$ be of class $\mathcal{S}_{1,\mu}$ ($\mu \geq 2$) and $\{W_h\}$ of class $\mathcal{S}_{1,\mu'}$ ($\mu' \geq 2$) and suppose that (25), (26), (27) and (28) hold for some ν, ν', with $\nu < \mu$ and $\nu' < \mu'$. Then, if condition (18) holds and the solution $\mathbf{u}$ is such that $u_1 \in H^1(0, T; H^\mu(\Omega))$ and $u_2, u_3 \in$*

$H^1(0,T;H^{\mu'}(\Omega))$, there exists an h_0 such that, for $h \le h_0$, the solution $\mathbf{u}_h$ of (12) exists for $t \le T$, and for these t we have

$$\|\mathbf{u}_h - \mathbf{u}\|_0 \le c\; h^{\min(\mu,\mu')}.$$

Proof. Let t^h be the largest number less than or equal to T such that $\mathbf{u}_h$ exists and belongs to Σ_δ for $t \le t^h$, i.e. $t^h = \sup\{s \le T \;:\; \mathbf{u}_h(t) \in \Sigma_\delta \;\forall t \le s\}$.

Similarly to Theorem 2.2, we get the estimate (24) for $t < t^h$ and, taken into account the assumptions on V_h and W_h and the smoothness of $\mathbf{u}$, we have

$$\|(u_{1,h}(t), u_{2,h}(t), u_{3,h}(t)) - (u_1(t), u_2(t), u_3(t))\|_0 \le ch^{\tilde{\mu}},\;\; \tilde{\mu} = \min(\mu,\mu'). \tag{29}$$

Moreover, for $t < t^h$, we have

$$\|\mathbf{u}_h - \mathbf{u}\|_\infty \le \|u_{1,h} - u_1\|_\infty + \|u_{2,h} - u_2\|_\infty + \|u_{3,h} - u_3\|_\infty \tag{30}$$

Let us consider the first term in the right hand side of (30). Let w_h be an arbitrary element of V_h; from (25), it follows

$$\|u_{1,h} - u_1\|_\infty \le \|u_{1,h} - w_h\|_\infty + \|w_h - u_1\|_\infty \le Ch^{-\nu}\|u_{1,h} - w_h\|_0 + \|w_h - u_1\|_\infty \le$$

$$\le Ch^{-\nu}\|u_{1,h} - u_1\|_0 + Ch^{-\nu}\|w_h - u_1\|_0 + \|w_h - u_1\|_\infty$$
$$\le ch^{\mu-\nu} + C \inf_{w_h \in V_h} \{h^{-\nu}\|u_1(t) - w_h\|_0 + \|u_1(t) - w_h\|_\infty\} < \delta/6 \qquad h \le h_1,$$

since $\nu < \mu$ and (27) holds. Similar results can be found for the other two terms in (30). Therefore

$$\forall t < t^h. \qquad \|\mathbf{u}_h(t) - \mathbf{u}(t)\|_\infty < \delta/2, \qquad h \le h_0 \tag{31}$$

where h_0 is sufficiently small and independent of t^h. Hence we may conclude by continuity that t^h cannot be smaller than T, that is $t^h = T$ for $h \le h_0$ and, by (29),

$$\|\mathbf{u}_h(t) - \mathbf{u}(t)\|_0 \le ch^{\tilde{\mu}} \qquad \forall t \le T. \;\square$$

Remark 2.6 *In particular, estimate (31) yields the boundedness of the semidiscrete solution $\mathbf{u}_h$.*

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